NOTES ON THE USE OF DIMENSIONAL ANALYSIS IN PSYCHOPHYSICS

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Abstract

It is argued that Dimensional Analysis (from physics) may be used in psychophysics as efficiently as in physics, provided the dimensions of some non-physical variables are interpreted in the sense of Reversion Dimensional Analysis - a method which allows for extracting (computing) dimensions from data.

The essence of Dimensional Analysis (DA), as thought of in physics, is that physical laws shall not depend on the choice of measurement scales and units. A physical equation shall be dimensionally homogeneous, i.e. its left and right sides shall have the same physical dimension(s). DA has proven its efficiency of a powerful analytical tool in almost all branches of physics as well as in disciplines such as engineering and chemistry (Barenblatt, 1987; Bridgman, 1922; Sedov, 1981). The method is used from simple error-check of physical equations, to obtaining explicit functional relations among variables relevant to a phenomenon being investigated. Limited attempts of applying DA have been made in other fields such as economics (De Jong, 1967) and physical geography (Haynes, 1982). With respect to psychology, DA has enjoyed certain attention on the part of the measurement theoreticians, primarily due to the works of R.D. Luce, however in rather purely theoretical aspect (cf. Krantz et. al., 1971 and references therein). Perhaps this is the reason why D. Laming recently wrote "While the argument from DA might be justifiable as an exercise in axiomatization, it contributes nothing to psychophysics" (Laming, 1997, p.12). It is not difficult to agree with this author, for in order to apply DA to certain problem one shall know the dimensions of all variables involved. While this is not an issue in physical applications, it appears to be a fundamental obstacle beyond physics, as some of the variables do not have dimensions in the sense it is thought of in physics. The problem is therefore not the limited power of DA per se, but the lack of clear understanding, if any, of what a dimension of a non-physical variable is. A possible solution to this problem was suggested by the author of this article (Marinov, 1994, 1995, 1999, 2000; Marinov & Panov, 2000). The essence of this approach, referred to as Reversed Dimensional Analysis (RDA), is that if a dimension cannot be assigned to a variable in the way it takes place in physics, but such a variable is yet measurable, then such a dimension can be extracted (literally calculated) from data. One can immediately realize that such a concept of dimension (quantity a posteriori computed from data) is fundamentally different from the classical concept of dimension in physics.
(qualitative denotation a priori assigned to a physical property). The author himself was initially confused by the novelty of this concept until he had a chance to read a monograph on fractal geometry (Mandelbrot, 1983) and realized that a fractal dimension, a concept already recognized by the scientific community, is in fact a particular case of a dimension obtained by RDA when only spatial (geometrical) properties of an object/event are considered.

Hereafter we discuss some examples of possible applications of DA to psychophysics, and in analysis of psychological data in general, provided the dimensions of some non-physical variables are treated in the sense of RDA.

Consider Stevens’ psychophysical law $S = C \cdot I^n$, where $S$ is the magnitude of sensation, $I$ is the stimulus intensity, and $C$ and $n$ are parameters obtained via regression of experimental data. Without losing generality, consider estimation of time intervals. In such a case $S$ is the response of the subject either in terms of time duration (seconds, minutes, etc.), or in terms of dimensionless numbers. $I$ is the stimulus duration in certain physical units of time. In other words, the dimension of $S$ is either $[S] = T$ (dimension of time) or $[S] = 1$ (dimensionless), while the dimension of $I$ is always $[I] = T$, where brackets are used to denote the dimension of the variable. Respectively, in order to maintain the dimensional homogeneity of the law, the dimension of $C$ shall be either $[C] = T^{-n+1}$ or $[C] = T^{-n}$. The commonly accepted value of the exponent in the Stevens’ law for time duration is $n = 1.15$ (Stevens, 1959, Luce & Krumhansl, 1988). The first two rows of Table 1 summarize what was just said.

**Table 1**

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>$C$</td>
<td>$I$</td>
</tr>
<tr>
<td>$T$</td>
<td>$T^{-n+1}$</td>
<td>$T$</td>
</tr>
<tr>
<td>$C$</td>
<td>$T^n$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T^n$</td>
<td>$1$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

Case 1 and Case 2 correspond to experimental conditions when the subject is asked to estimate time intervals in terms of time duration (seconds, minutes, etc.) or in terms of dimensionless numbers, respectively. Case 3 is essentially different. Obviously, the subject cannot be asked to estimate time intervals in terms of a quantity which dimension is a priori unknown. Such a dimension can however be calculated a posteriori from data by the method of RDA briefly described above. Practically, the data can be obtained in the way represented by Case 2, but then the variable $S$ to be treated as a quantity of unknown dimension and the dimension calculated via RDA. One can see that the three cases do no differ in the sense that a constant/variable of unusual and difficult to interpret dimension is present. Then, what is the advantage of the approach represented by Case 3?

The dimension of a quantity is thought of as something more fundamental than the form of the functional relation the quantity is involved in. Therefore, while the functional relation may change from task to task, it may be expected that the dimension of the quantity will not. In fact, except for some exotic cases (e.g., dimensional transmutation (cf., Collins, 1984)), the
entire science of physics is an example of dealing with quantities which preserve their dimensions while enter different functional relations. Thus, with respect to time, may we expect that dimensions obtained via RDA in different psychological tasks will have the same numerical values? Apart from statistical variations, the answer is indeed not trivial. Some researchers favor a poly-chronic view of the world with many different scales of time, in which time itself is created by system activity (cf., Ward, 2001). If this is the case, then the dimensions in question may not necessarily have the same numerical value when different tasks are performed. If, however, processing of time is scale invariant*, then psychological variables related to time stimuli may exhibit a characteristic dimension of the same, or very close, numerical value throughout a variety of tasks. Vice versa, if dimensions obtained in different tasks essentially differ, then it might be concluded that those particular tasks are not scale invariant with respect to time.

In order to illustrate the above point, consider the data of Hellyer (1962) discussed in detail in Marinov & Panov (2000). In a Peterson & Peterson’ type of experiment (see, for instance, Laming (1992)) Hellyer studied the Proportion of items (consonant trigrams) correctly recalled (X1) as a function of two variables - Recall delay interval (X2) and Number of presentations (X3). The relation between these three variables can be expressed as

\[ X_1 = f(X_2, X_3) \]

where, according to Hellyer, the dimensions of the variables are: \([X_1] = 1\) (dimensionless), \([X_2] = T\) (time) and \([X_3] = 1\) (dimensionless). Apparently, this relation is dimensionally inhomogeneous unless at least one dimensional constant, say \(A\), with a dimension of the kind \(T^m\) is introduced. The exponent \(m\) can be determined from data via regression in exactly the same way the Stevens’ exponent \(n\) is determined. Thus, an analogue of Table 1, except for Case 1, is Table 2 below.

<table>
<thead>
<tr>
<th>Case</th>
<th>(X_1)</th>
<th>(A)</th>
<th>(X_2)</th>
<th>(X_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>(T^{-m})</td>
<td>(T)</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>(T)</td>
<td>(T^n)</td>
</tr>
</tbody>
</table>

* Scale invariance is closely related to self-similarity. Both concepts are best illustrated in terms of spatial properties of objects. An object is self-similar if every part of the object, when magnified by a suitable scale factor, is identical to the original object. Scale invariance is a related and stronger property, which is thought of as self-similarity on all scales (Creswick, Farach & Poole, 1992). The concepts can be generalized with respect to other properties of objects/phenomena.
Thus, in two essentially different experiments in the field of human psychology - magnitude estimation and memory recall - it was found that variables related to processing of time exhibit practically the same numerical value of their dimensions when computed by the method of RDA - \( n=1.15 \) and \( n=1.13 \), respectively. Is this always the case, so that we could claim that there exists a universal mental scale in processing of time, or at least that processing of time is scale invariant? Currently, the limited number of psychological data processed by RDA (cf., Marinov 1994, 1995, 2000; Marinov, Bojanska & Kohut, 1994; Marinov & Panov, 2000) does not really allow to give a definitive answer. Yet, apart from the almost identical values just discussed, there is some evidence that RDA produces rather similar results for data obtained in similar conditions in the sense of processing time. On the other hand, it allows to clearly distinguish data obtained in seemingly similar conditions, which in fact differ essentially. Figure 1, adapted from Marinov (2000), illustrates this statement.

![Figure 1](image1)

**Figure 1.** Top: The Coefficient of variation \( CV \) as a function of the exponent \( p \) (Peterson & Peterson’s data). Bottom: The Relative Coefficient of variation \( CV/CV_{min} \) as a function of the exponent \( p \) (Peterson & Peterson’s and Hellyer’s data). Interval of seeking minima: \(-10 \leq p \leq +10\). Grid step: \( \Delta p = 0.001 \). The curves appear (from left to right) as in the legend (from top to bottom).

Avoiding lengthy discussions on RDA technicalities, in order to clarify Figure 1, we just note that the method is statistical in nature. The unknown dimension is computed by minimizing certain statistical parameter, in this case the coefficient of variation for residuals (\( CV \)), as a function of the dimension \( p \), which is considered as a continuous variable. The minimum \( CV_{min} = CV(p_{min}) \), if such minimum exists, gives the numerical value of the dimension, provided an *a priori* stated statistical criterion of the kind \( CV_{min} \leq \delta \) is met. This statement also implies that the computed dimension shall be interpreted not as an exact number but rather as an interval \( \hat{p} \leq p \leq \hat{p}^b \) specified by a condition of the kind \( (CV - CV_{min})/CV_{min} \leq \epsilon \), where \( \epsilon \) is an *a priori* chosen small number. In other words, the smaller \( CV_{min} = CV(p_{min}) \) and the narrower the U-curve, the more precise the determination of the unknown dimension.
The first observation to be noticed is that all four curves in Figure 1, corresponding to four different data sets, are U-shaped downward. In other words, \( p_{\text{null}} \) exists. This is not a trivial corollary of the method of RDA per se. To the contrary, model experiments show that there exist cases when \( CV(p) \) does not have minimum within intervals of seeking minima as large as \(-2 \leq p \leq 2\). The second observation to be noticed is that data corresponding to similar conditions produce U-curves which are similarly shaped and closely grouped together. As argued in Marinov (2000) the “vocal” and the “letter” data sets reported by Peterson & Peterson (1959), as well as the data set reported by Hellyer (1962), correspond to similar conditions in the sense that the variable \( \text{Number of presentations} (X_3) \), can be meaningfully interpreted as a variable with a dimension of time. The computed values of the dimensions are as follows: P&P “vocal” - \( T^{1.55} \), P&P “letter” - \( T^{1.35} \), and Hellyer – \( T^{1.13} \). To the contrary, the “silent” data set reported by Peterson & Peterson (1959) along with the other two sets, produces dramatically different U-curve (the leftmost curve in Figure 1). The computed dimension in this case is \( T^{4.82} \), which is essentially different from those obtained from the other three data sets. Moreover, the widely stretched U-curve does not really allow any specific determination of a dimension. As argued in Marinov (2000), this is because for this particular data set the variable \( \text{Number of presentations} (X_3) \) cannot be meaningfully interpreted as a variable with a dimension of time.

Thus, based on analysis of a rather limited quantity of data by means of RDA, it seems reasonable to claim that data obtained in essentially different psychological experiments exhibit close numerical values of computed dimensions, provided the corresponding variables allow for meaningful interpretation in terms of time intervals. This finding may be interpreted in the sense that either a universal time scale exists or processing of time by humans is scale invariant.

Concluding this rather qualitative explanation of our view on possible applications of Dimensional Analysis (DA) in psychophysics, we emphasize that DA appears as promising and efficient as it is in physics, provided the dimensions of some non-physical variables are interpreted in the sense of Reversed Dimensional Analysis (RDA).

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